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Higher dimensional spacetimes with a geodesic, shearfree, twistfree and expanding null congruence*

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ABSTRACT: We present the complete family of higher dimensional spacetimes that admit a geodesic, shearfree, twistfree and expanding null congruence, thus extending the well-known $D = 4$ class of Robinson-Trautman solutions. Einstein's equations are solved for empty space with an arbitrary cosmological constant and for aligned pure radiation. Main differences with respect to the $D = 4$ case (such as the absence of type III/N solutions, related to “violations” of the Goldberg-Sachs theorem in $D > 4$) are pointed out, also in connection with other recent works. A formal analogy with electromagnetic fields is briefly discussed in an appendix, where we demonstrate that multiple principal null directions of null Maxwell fields are necessarily geodesic, and that in $D > 4$ they are also shearing if expanding.

Dedicated to Massimo Pauri on the occasion of his nomination as Professor Emeritus.

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1. Introduction

Thanks to the correspondence between geometrical properties of null geodesics and optical properties of the gravitational field, the study of ray optics [1–5] has played a major role in the construction and classification of exact solutions of Einstein's equations [6]. This applies in particular to solutions representing gravitational radiation. Remarkably, the fundamental families of “spherical” and “plane-fronted” gravitational waves found, respectively, by Robinson and Trautman [7, 8] and by Kundt [9, 10] are invariantly defined in terms of geometric optics. Relying on formal analogies with the theory of electromagnetic radiation, Robinson-Trautman solutions are characterized by the existence of a geodesic, non-twisting, non-shearing and expanding null congruence, whereas Kundt metrics (which include pp -waves) require a non-twisting and non-shearing congruence but with vanishing expansion. In both cases, the shear-free condition implies that the corresponding spacetimes are algebraically special (at least in vacuum or in the presence of “sufficiently aligned” matter), due to the Goldberg-Sachs theorem [3, 11]. In fact, explicit solutions of all special Petrov types are known within both classes of geometries [6–10].

The geometric optics approach was naturally developed in $D = 4$ General Relativity. On the other hand, in recent years gravity in more than four spacetime dimensions has become an active area of ongoing interdisciplinary studies. It is thus now interesting

to investigate possible extensions of the above concepts to arbitrary dimensions $D > 4$. Certain properties of higher dimensional Kundt solutions have been analyzed in [12–15] (these in particular contain all spacetimes with vanishing scalar curvature invariants [12, 15, 16]). It is the purpose of our contribution to summarize the systematical derivation of the $D > 4$ Robinson-Trautman class of solutions performed in [17] and to discuss the main features of such spacetimes and recent related developments. We shall focus here on the case of vacuum spacetimes, with a possible cosmological constant, and of aligned pure radiation. All our results will be local.

Aspects of geometric optics in higher dimensions have been also studied in [18–20], and “violations” of the standard $D = 4$ Goldberg-Sachs theorem in $D > 4$ have been pointed out. For example, the principal null directions of $D = 5$ rotating black holes (which are of type D) are geodesic but shearing [18, 19]. Furthermore, in $D > 4$ vacuum spacetimes of type III or N, a multiple principal null direction with expansion necessarily has also non-zero shear (still being geodesic) [19]. This implies in particular that there do not exist shearfree, twistfree and expanding (i.e., Robinson-Trautman) vacuum solutions of type III or N when $D > 4$, as opposed to the $D = 4$ case [6–8]. Our results, obtained with a method different from that of [19], will be in agreement with this conclusion. Furthermore, we shall point out that also electromagnetic “null” fields share similar properties in $D > 4$.

The paper is organized as follows. In section 2 we present the general form of the Robinson-Trautman line element under purely geometrical requirements on the optical scalars. In section 3 we derive the explicit solution to Einstein’s equations within such a setting. In section 4 we discuss general features of the obtained spacetimes, such as the algebraic structure of the corresponding Weyl tensor, whereas we concentrate on the special case of vacuum solutions in section 5. We present concluding remarks in section 6. Appendix A summarizes geometric optics definitions in higher dimensions, mainly following [19] (cf. also [18, 20]). Appendix B studies optical properties of higher dimensional null Maxwell fields and shows that their rays are necessarily geodesic, and that they must be shearing if expanding.

2. Geometrical assumptions: spacetimes with a geodesic, shearfree and twistfree null congruence

Let us consider a generic D -dimensional spacetime ($D \geq 4$). A null congruence with tangent vector field \tilde{k}^α is (locally) orthogonal to a family of null hypersurfaces $u(x) = \text{const}$ (i.e., $\tilde{k}_\alpha = -f(x)u_{,\alpha}$, with $g^{\alpha\beta}u_{,\alpha}u_{,\beta} = 0$) if and only if $\tilde{k}_{[\alpha;\beta}\tilde{k}_{\rho]} = 0$.¹ The latter condition, in turn, is equivalent to \tilde{k}^α being geodesic and twistfree, in the sense that the twist matrix of [19] vanishes (see appendix A). Given such a congruence, the rescaled tangent field $k_\alpha = -u_{,\alpha}$ will also be null, twistfree and geodesic and, in addition, affinely parameterized (i.e., $k_{\alpha;\beta}k^\beta = 0$). Now, it is natural to take the function u itself (constant along each ray) as one of the coordinates, so that $k_\alpha = -\delta_\alpha^u$ and $g^{uu} = 0$. As for the remaining coordinates, we use the affine parameter r along the geodesics generated by k^α , and “transverse” spatial

¹The implication $\tilde{k}_\alpha = -f(x)u_{,\alpha} \Rightarrow \tilde{k}_{[\alpha;\beta}\tilde{k}_{\rho]} = 0$ is obvious. The converse follows from the Frobenius theorem, see e.g. [6].

coordinates $(x^2, x^3, \dots, x^{D-1})$ which are constant along these null geodesics (cf. [8, 21]). This further implies $k^\alpha = \delta_r^\alpha$, that is $g^{ur} = -1$ and $g^{uu} = 0 = g^{ui}$. Therefore, the covariant line element can be written as

$$ds^2 = g_{ij} (dx^i + g^{ri} du) (dx^j + g^{rj} du) - 2dudr - g^{rr} du^2, \quad (2.1)$$

where the metric coefficients are, for now, arbitrary functions of all the coordinates (x, u, r) (hereafter, x stands for all the transverse coordinates x^i and lowercase latin indices range as $i = 2, \dots, D-1$). Useful relations between the covariant and contravariant metric coefficients, to be employed in the sequel, are

$$g^{ri} = g^{ij} g_{uj}, \quad g^{rr} = -g_{uu} + g^{ij} g_{ui} g_{uj}, \quad g_{ui} = g^{rj} g_{ij}, \quad (2.2)$$

while $g_{rr} = 0 = g_{ri}$. In this coordinate system, it is also easy to see that

$$k_{\alpha;\beta} = \frac{1}{2} g_{\alpha\beta,r}. \quad (2.3)$$

For later purposes, it is convenient (cf. [8, 17]) to define an auxiliary $(D-2)$ -dimensional spatial metric γ_{ij} by

$$\gamma_{ij} = p^2 g_{ij}, \quad p^{2(2-D)} \equiv \det g_{ij} = -\det g_{\alpha\beta}, \quad (2.4)$$

so that $\det \gamma_{ij} = 1$. Then one can express the generalized optical scalars expansion and shear [18, 19] (cf. appendix A) associated to k^α simply as [17]²

$$\begin{aligned} \text{Tr}(\sigma^2) &\equiv k_{(\alpha;\beta)} k^{\alpha;\beta} - \frac{1}{D-2} (k^\alpha_{;\alpha})^2 = \frac{1}{4} \gamma^{li} \gamma^{kj} \gamma_{ki,r} \gamma_{lj,r}, \\ \theta &\equiv \frac{1}{D-2} k^\alpha_{;\alpha} = -(\ln p)_{,r}. \end{aligned} \quad (2.5)$$

Now, imposing the condition that the congruence k^α is *shear-free*, $\text{Tr}(\sigma^2) = 0$, eq. (2.5) leads to

$$\gamma_{ij,r} = 0, \quad (2.6)$$

since there always exists a frame in which γ^{ij} is diagonal, with strictly positive eigenvalues.

In summary, the line element of any spacetime admitting a hypersurface orthogonal (i.e., geodesic and twistfree) shear-free null congruence $k = \partial_r$ can be written in the form (2.1), with $g_{ij} = p^{-2} \gamma_{ij}$; the matrix γ_{ij} is unimodular and independent of r , while p , g^{ri} and g^{rr} are arbitrary functions of (x, u, r) . Note that such a metric is left invariant by the following coordinate transformations (which do not change the family of null hypersurfaces $u = \text{const}$ nor the affine character of the parameter r)

$$x^i = x^i(\tilde{x}, \tilde{u}), \quad u = u(\tilde{u}), \quad r = r_0(\tilde{x}, \tilde{u}) + \tilde{r}/\dot{u}(\tilde{u}). \quad (2.7)$$

²The definitions of the scalars $\text{Tr}(\sigma^2)$ and θ in (2.5) hold only when an *affine* parameter is used along k^α .

3. Integration of Einstein's equations

After deriving the above line element (2.1) (with eq. (2.4)), one has to integrate Einstein's equations $R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta}$ to determine the unknown metric coefficients g_{ij} , g^{ri} and g^{rr} . Here we concentrate on the case of vacuum spacetimes ($T_{\alpha\beta} = 0$) and of aligned pure radiation ($T_{\alpha\beta} = \Phi^2 k_\alpha k_\beta$), while the cosmological constant Λ is arbitrary. In the coordinate system introduced above this means that only the $T_{uu} = \Phi^2$ component can be non-vanishing. Then, recalling the line element (2.1) and the relations (2.2), Einstein's equations effectively reduce to the following set of equations: $R_{rr} = 0$, $R_{ri} = 0$, $R_{ij} = \frac{2}{D-2}\Lambda g_{ij}$, $R_{ur} = -\frac{2}{D-2}\Lambda$, $R_{ui} = \frac{2}{D-2}\Lambda g_{ui}$ and $R_{uu} = \frac{2}{D-2}\Lambda g_{uu} + 8\pi\Phi^2$.

Since we have assumed that twist and shear are identically vanishing, the generalized Sachs equation governing the rate of change of the expansion [17,20] (namely $(D-2)\theta_{,\alpha}k^\alpha + \text{Tr}(\sigma^2) + (D-2)\theta^2 + \text{Tr}(A^2) = -R_{\alpha\beta}k^\alpha k^\beta$) implies that for the R_{rr} component we have simply $R_{rr} = -(D-2)(\theta_{,r} + \theta^2)$. Hence the integration of $R_{rr} = 0$ singles out two alternative solutions. The first arises when $\theta = 0$ which, by (2.5), is equivalent to $p = p(x, u)$. This possibility corresponds to the Kundt class of non-expanding spacetimes, considered in [12–15] for $D > 4$. But we are here interested in the alternative case $\theta \neq 0$ of an expanding vector field k^α , for which one finds $\theta^{-1} = r + r_0(x, u)$. The arbitrary function r_0 can always be set to zero by a transformation (2.7), so that

$$\theta = \frac{1}{r}. \quad (3.1)$$

From eq. (2.5) we can thus factorize $p = r^{-1}P(x, u)$, where P is an arbitrary function. It is also convenient to rescale the transverse metric by introducing $h_{ij} = P^{-2}\gamma_{ij}$, so that the relation (2.4) becomes

$$g_{ij} = p^{-2}\gamma_{ij} = r^2P^{-2}\gamma_{ij} = r^2h_{ij}(x, u). \quad (3.2)$$

This specifies the r -dependence of g_{ij} in the metric (2.1). Note that $P^2 = (\det h_{ij})^{1/(2-D)}$.

One should now proceed to integrate all the remaining Einstein equations. Since this is lengthy, we just summarize here the main results and refer to [17] for technical details. First, imposing (a subset of) the Einstein equations and using an appropriate coordinate transformation one can always set

$$g^{ri} = 0. \quad (3.3)$$

Then, one finds that at any given $u = u_0 = \text{const}$ each $(D-2)$ -dimensional spatial metric $h_{ij}(x, u_0)$ must be an Einstein space (in $D = 5$ this implies that h_{ij} is a 3-space of constant curvature); also, the u -dependence of h_{ij} must be factorized out in a conformal factor. Namely,

$$\mathcal{R}_{ij} = \frac{\mathcal{R}}{D-2}h_{ij}, \quad (3.4)$$

$$h_{ij} = P^{-2}(x, u)\gamma_{ij}(x), \quad \text{where} \quad \det \gamma_{ij} = 1, \quad (3.5)$$

in which \mathcal{R}_{ij} is the Ricci tensor associated with the metric h_{ij} , and \mathcal{R} the corresponding Ricci scalar. As a well-known fact, any two-dimensional metric h_{ij} satisfies eq. (3.4), so

that this is identically satisfied in the special case $D = 4$. Therefore for $D = 4$ eq. (3.4) puts no restriction on h_{ij} (in particular, the scalar curvature \mathcal{R} of h_{ij} can generally depend both on u and on the spatial coordinates x). On the other hand, for any $D > 4$, eq. (3.4) tells us (via the contracted Bianchi identities) that \mathcal{R} can not depend on the x coordinates, i.e.

$$\mathcal{R} = \mathcal{R}(u) \quad (D > 4). \quad (3.6)$$

As a consequence, the next equation to be solved (which controls the u -dependence of P) differs crucially in the $D = 4$ and $D > 4$ cases, namely one has

$$\frac{1}{2}(\mathcal{R}_{,i}h^{ij})_{,j} - (\mathcal{R}_{,i}h^{ij})(\ln P)_{,j} + 6\mu(\ln P)_{,u} - 2\mu_{,u} = 16\pi n^2 \quad (D = 4), \quad (3.7)$$

$$(D - 1)\mu(\ln P)_{,u} - \mu_{,u} = \frac{16\pi n^2}{D - 2} \quad (D > 4), \quad (3.8)$$

where $n = n(x, u)$ and $\mu = \mu(u)$ are arbitrary functions. The former characterizes the pure radiation term, which must take the form

$$T_{uu} = \Phi^2 = r^{2-D}n^2(x, u), \quad (3.9)$$

whereas the latter enters the last metric coefficient $g^{rr} = -g_{uu} \equiv 2H$, given by

$$2H = \frac{\mathcal{R}}{(D - 2)(D - 3)} - 2r(\ln P)_{,u} - \frac{2\Lambda}{(D - 2)(D - 1)}r^2 - \frac{\mu(u)}{r^{D-3}}. \quad (3.10)$$

Note that the first two terms in eq. (3.7) represent (one-half of) the covariant Laplace operator on a 2-space with metric h_{ij} (applied to \mathcal{R}). Renaming $\mu = 2m$ we thus obtain $\frac{1}{2}\Delta\mathcal{R} + 12m(\ln P)_{,u} - 4m_{,u} = 16\pi n^2$, which is the standard form of the Robinson-Trautman equation [6]. We will not discuss the case $D = 4$ any longer here since it is already well-known.

4. Properties of the solutions in $D > 4$

To summarize, the Robinson-Trautman class of solutions in $D > 4$ with aligned pure radiation, as obtained above, reads

$$ds^2 = r^2 P^{-2} \gamma_{ij} dx^i dx^j - 2du dr - 2H du^2, \quad (4.1)$$

with eqs. (3.2), (3.4), (3.5), (3.8), (3.9) and (3.10). Notice that using the coordinate freedom remaining from (2.7), namely the reparametrization of u ($\dot{u} > 0$),

$$u = u(\tilde{u}), \quad r = \tilde{r}/\dot{u}(\tilde{u}), \quad \text{so that} \quad \tilde{P} = P\dot{u}, \quad \tilde{\mathcal{R}} = \mathcal{R}\dot{u}^2, \quad \tilde{\mu} = \mu\dot{u}^{D-1}, \quad \tilde{n}^2 = n^2\dot{u}^D, \quad (4.2)$$

($\tilde{\mathcal{R}}$ is indeed the Ricci scalar of the rescaled metric $\tilde{h}_{ij} = \dot{u}^{-2}h_{ij}$) we may achieve further useful simplification of the metric (4.1), (3.10). For example, we can always put $\tilde{\mu}$ or (in $D > 4$) $\tilde{\mathcal{R}}$ to be a constant. Also, transformations of the coordinates $x^i = x^i(\tilde{x})$ can be used to change the form of the spatial metric h_{ij} . In any case, for fixed r and u the metric \tilde{h}_{ij} can be any Riemannian Einstein space (see eq. (3.4)), the u -dependence of this family

being governed solely by P . The variety of possible metrics h_{ij} is thus huge. For example, in addition to the simplest case when h_{ij} is a space of constant curvature, if $\mathcal{R} > 0$ and $5 \leq D-2 \leq 9$ one can take any of the infinite number of non-trivial compact Einstein spaces presented in [22].

One can also compute the Weyl tensor corresponding to the metric (4.1). Its non-zero components read [17]

$$\begin{aligned} C_{ruru} &= -\mu(u) \frac{(D-2)(D-3)}{2r^{D-1}}, & C_{riu j} &= \mu(u) \frac{(D-3)}{2r^{D-3}} h_{ij}, \\ C_{ijkl} &= r^2 \mathcal{R}_{ijkl} - 2r^2 \left(\frac{\mathcal{R}(u)}{(D-2)(D-3)} - \frac{\mu(u)}{r^{D-3}} \right) h_{i[k} h_{l]j}, \\ C_{uiuj} &= 2H C_{riu j}. \end{aligned} \quad (4.3)$$

Using a suitable frame based on the null vectors

$$\mathbf{k} = \partial_r, \quad \mathbf{l} = -\partial_u + H\partial_r \quad (4.4)$$

(such that $\mathbf{k} \cdot \mathbf{l} = 1$), it is straightforward to see that the above coordinate components give raise only to frame components of boost weight zero. The Weyl tensor is thus of type D (unless vanishing) in the classification of [23, 24]. This should be contrasted with the case $D = 4$, in which all algebraically special types are allowed. Type O (conformally flat) spacetimes are possible only if $\mu = 0$, which implies a vanishing pure radiation field (cf. eq. (3.8)) and therefore that only constant curvature spacetimes can occur in this case.

5. Vacuum solutions

The special case of vacuum Robinson-Trautman spacetimes is given by $T_{uu} = 0$, i.e. $n = 0$ in eq. (3.8). In this case, one has to consider separately the two cases $\mu \neq 0$ and $\mu = 0$.

5.1 Case $\mu \neq 0$

When $\mu \neq 0$, it can always be set to a constant by a rescaling (4.2). Eq. (3.8) thus reduces to $\mu P_{,u} = 0$. Consequently, in this case the function P must be independent of u , and thus must therefore be also h_{ij} . It follows, in particular, that \mathcal{R} is a constant. Unless now $\mathcal{R} = 0$, one can choose the transformation (4.2) to normalize $\mathcal{R} = \pm(D-2)(D-3)$. The corresponding Robinson-Trautman geometries are thus fully characterized by the line element (4.1) with the simple function

$$2H = K - \frac{2\Lambda}{(D-2)(D-1)} r^2 - \frac{\mu}{r^{D-3}} \quad (K = 0, \pm 1), \quad (5.1)$$

and they clearly admit ∂_u as a Killing vector. As mentioned above, the spatial metric $h_{ij} = P^{-2}(x) \gamma_{ij}(x)$ can describe *any* Einstein space with scalar curvature $\mathcal{R} = K(D-2)(D-3)$. When this space is compact, such a family of solutions describes various well-known static black holes in Eddington-Finkelstein coordinates. In particular, if the horizon has constant

curvature one obtains Schwarzschild-Kottler-Tangherlini black holes [25], for which the line element can always be cast in the form [17]

$$ds^2 = r^2 \left(1 + \frac{1}{4}K\delta_{kl}x^kx^l\right)^{-2} \delta_{ij}dx^i dx^j - 2dudr - 2Hdu^2. \quad (5.2)$$

In addition, there are generalized black holes [26–28] with various horizon geometries (e.g., those presented in [22] for $K = +1$), non-standard asymptotics and, possibly, non-spherical horizon topology. As we have seen in section 4, all these solutions are of type D, and the components (4.3) of the Weyl tensor become

$$\begin{aligned} C_{ruru} &= -\mu \frac{(D-2)(D-3)}{2r^{D-1}}, & C_{riu j} &= \mu \frac{(D-3)}{2r^{D-3}} h_{ij}, \\ C_{ijkl} &= r^2 \mathcal{R}_{ijkl} - 2r^2 \left(K - \frac{\mu}{r^{D-3}}\right) h_{i[k} h_{l]j}, \\ C_{uiuj} &= 2H C_{riu j}. \end{aligned} \quad (5.3)$$

5.2 Case $\mu = 0$

In the exceptional case $\mu = 0$, eq. (3.8) is identically satisfied in vacuum, and one can not conclude that P is independent of u . One can still rescale $\mathcal{R}(u)$ to be a constant $\mathcal{R} = K(D-2)(D-3)$, $K = 0, \pm 1$, so that in this case the line element (4.1) contains the characteristic function

$$2H = K - 2r(\ln P)_{,u} - \frac{2\Lambda}{(D-2)(D-1)} r^2. \quad (5.4)$$

Now the only non-vanishing components of the type D Weyl tensor are

$$C_{ijkl} = r^2 (\mathcal{R}_{ijkl} - 2K h_{i[k} h_{l]j}). \quad (5.5)$$

These spacetimes degenerate to type O (thus to constant curvature since vacuum) when $C_{ijkl} = 0$, which is equivalent to having a transverse spatial metric h_{ij} of constant curvature K . In particular, this is necessarily the case in $D = 5$ (cf. section 3).

Now, recall from eq. (3.5) that $h_{ij}(x, u) = P^{-2}(x, u) \gamma_{ij}(x)$, i.e. the possible u -dependence of the line element (4.1) is contained only in P . In the simplest case of a factorized function $P(x, u) = P_1(x)P_2(u)$, this u -dependence is obviously removable with a transformation (4.2). The corresponding solutions are therefore equivalent to the metrics of subsection 5.1 with $\mu = 0$. In general, when the u -dependence is not factorized, the solutions (5.4) are instead presented in a somewhat implicit form, in the sense that one has still to specify an Einstein metric h_{ij} such that it depends on u only via an overall conformal factor. Finding an explicit spacetime of the type (5.4) with a non-trivial u -dependence (in $D > 5$, since the trivial case $D = 5$ has been already discussed above) could be in principle all but straightforward. Notice, however, the following: if we think of h_{ij} as a family of $(D-2)$ -dimensional Riemannian Einstein metrics parametrized by u , it is obvious that all such Einstein spaces are conformally related. We can thus take any known Einstein space and obtain from it a conformally related Einstein space with an appropriate u -dependence in the conformal factor. In fact, conformal mapping of Einstein spaces on Einstein spaces

were studied thoroughly by Brinkmann [21, 29] (see also, e.g., [30, 31]). In particular, four-dimensional Riemannian Einstein spaces which admit a conformal (non-homothetic) map on Einstein spaces must be of constant curvature [21, 29]. Therefore, also in $D = 6$ (which means that h_{ij} is four-dimensional) solutions (5.4) trivially reduce to constant curvature spacetimes (Minkowski or (anti-)de Sitter), in view of eq. (5.5). On the other hand, in $D \geq 7$ we can employ the “canonical” form [21, 29–31] of Einstein spaces which can be mapped conformally on other Einstein spaces to construct an explicit, non-trivial solution of the form (5.4). For example, in $D = 7$ one of the simplest metrics we can think of is the following (we take $(z, \tau, \rho, \theta, \phi)$ as our (x^2, \dots, x^6) coordinates)

$$K = -1, \quad P(u, z, \rho, \theta) = f(u, z)^{-1/2} (\rho^2 \sin \theta)^{-1/5}, \quad (5.6)$$

$$h_{ij} dx^i dx^j = f(u, z) \left[dz^2 + \left(1 - \frac{2m}{\rho} - \frac{\rho^2}{l^2} \right) d\tau^2 + \left(1 - \frac{2m}{\rho} - \frac{\rho^2}{l^2} \right)^{-1} d\rho^2 + \rho^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (5.7)$$

$$f(u, z) = \frac{4b(u)e^{2z/l}}{l^2 [e^{2z/l} - b(u)]^2}, \quad (5.8)$$

where m and l are constants and $b(u) > 0$ is an arbitrary function.³ This u -dependence turns out to have a specific geometrical meaning [32] which invariantly characterizes the spacetime and more general solutions (5.4) (as opposed to solutions (5.1)). The associated Weyl tensor is non-zero and one can show, e.g., that $C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} = 48m^2 f^{-2} r^{-4} \rho^{-6}$. Note also that one can set $\Lambda = 0$, if desired.

6. Conclusions

We have presented the complete family of higher dimensional spacetimes that contain a hypersurface orthogonal, non-shearing and expanding congruence of null geodesics, and that satisfy Einstein’s equations with an arbitrary cosmological constant and aligned pure radiation. In particular, we have discussed vacuum solutions. There appear fundamental differences with respect to the standard $D = 4$ family of Robinson-Trautman solutions and, remarkably, $D > 4$ Robinson-Trautman spacetimes can be only of type D or O. This is in agreement with the previous results of Ref. [19], which proved (via a study of the Bianchi identities) that the multiple principal null congruence of $D > 4$ type N and type III vacuum spacetimes must have non-zero shear when it has non-zero expansion, so that type N and type III can not occur in the vacuum Robinson-Trautman class. For the case of non-twisting null congruences, we have generalized this conclusion to include a non-vanishing cosmological constant, and to prove that in fact any algebraic type different from D and O is forbidden for vacuum spacetimes with an expanding but non-shearing null congruence.

It is interesting to remark that $D = 4$ Robinson-Trautman solutions were originally constructed so as to model spherical gravitational radiation [7, 8]. They were thus required

³We clearly used the four-dimensional Euclidean Schwarzschild-de Sitter solution as a “seed” metric for the five-dimensional Einstein space (5.7), (5.8).

to share certain algebraic and optical properties with corresponding solutions of Maxwell's equations describing electromagnetic waves, in particular “null” fields. In contrast, it turned out that null (type N) gravitational fields are not possible for $D > 4$ vacuum Robinson-Trautman solutions. We show in appendix B that expanding null electromagnetic fields provide a natural counterpart to this in the higher dimensional Maxwell theory.

A. Optical scalars in higher dimensions

In a D -dimensional spacetime, one can introduce a frame of vectors $(\mathbf{k}, \mathbf{l}, \mathbf{m}^{(i)})$ (with $i = 2, \dots, D-1$) which satisfy the “orthonormality” relations

$$\mathbf{k} \cdot \mathbf{l} = 1, \quad \mathbf{m}^{(i)} \cdot \mathbf{m}^{(j)} = \delta_{ij}, \quad \mathbf{k} \cdot \mathbf{k} = \mathbf{l} \cdot \mathbf{l} = \mathbf{k} \cdot \mathbf{m}^{(i)} = \mathbf{l} \cdot \mathbf{m}^{(i)} = 0, \quad (\text{A.1})$$

so that \mathbf{k} and \mathbf{l} are null and the $\mathbf{m}^{(i)}$ are spacelike. On such a frame, the metric takes the simple form $g_{\mu\nu} = 2k_{(\mu}l_{\nu)} + \delta_{ij}m_{\mu}^{(i)}m_{\nu}^{(j)}$. Also, one can decompose the covariant derivative of the null vector k_{μ} as [19]

$$k_{\mu;\nu} = K_{11}k_{\mu}k_{\nu} + K_{10}k_{\mu}l_{\nu} + K_{1i}k_{\mu}m_{\nu}^{(i)} + K_{i1}m_{\mu}^{(i)}k_{\nu} + K_{i0}m_{\mu}^{(i)}l_{\nu} + K_{ij}m_{\mu}^{(i)}m_{\nu}^{(j)}, \quad (\text{A.2})$$

where, obviously,

$$\begin{aligned} K_{11} &\equiv k_{\mu;\nu}l^{\mu}l^{\nu}, & K_{10} &\equiv k_{\mu;\nu}l^{\mu}k^{\nu}, & K_{1i} &\equiv k_{\mu;\nu}l^{\mu}m^{(i)\nu}, \\ K_{i1} &\equiv k_{\mu;\nu}m^{(i)\mu}l^{\nu}, & K_{i0} &\equiv k_{\mu;\nu}m^{(i)\mu}k^{\nu}, & K_{ij} &\equiv k_{\mu;\nu}m^{(i)\mu}m^{(j)\nu}. \end{aligned} \quad (\text{A.3})$$

We are particularly interested in the case of a *geodesic* vector field k^{μ} . It follows from eq. (A.2) that $k_{\mu;\nu}k^{\nu} = K_{10}k_{\mu} + K_{i0}m_{\mu}^{(i)}$, so that k^{μ} is geodesic if and only if $K_{i0} = 0$. In addition, k^{μ} is affinely parametrized if also $K_{10} = 0$. When k^{μ} is geodesic the spatial matrix K_{ij} acquires a special meaning since it is then invariant under null rotations preserving k^{μ} (and it simply rescales with boost weight one under a boost in the \mathbf{k} - \mathbf{l} plane). It can thus be used to invariantly characterize geometric properties of the null congruence of integral curves of k^{μ} . To this effect, let us decompose K_{ij} into its tracefree symmetric part, its trace and its antisymmetric part as

$$K_{ij} = \sigma_{ij} + \theta\delta_{ij} + A_{ij}, \quad (\text{A.4})$$

where

$$\sigma_{ij} \equiv K_{(ij)} - \frac{\text{Tr } K}{D-2}\delta_{ij} = \sigma_{ji}, \quad \theta \equiv \frac{\text{Tr } K}{D-2}, \quad A_{ij} \equiv K_{[ij]} = -A_{ji}. \quad (\text{A.5})$$

We shall refer to σ_{ij} and A_{ij} as the *shear* and *twist* matrix, respectively, and to θ as the *expansion* scalar. Along with θ , one can now construct other scalar quantities out of $k_{\mu;\nu}$ which are invariant under null and spatial rotations with fixed k^{μ} , e.g. the shear and twist scalars given by $\text{Tr}(\sigma^2)$ and $\text{Tr}(A^2)$ (again, these simply rescales with boost weight one under a boost in the \mathbf{k} - \mathbf{l} plane). If in addition k^{μ} is affinely parametrized, i.e. $K_{10} = 0$,

one can express the thus defined optical scalars solely in terms of k^μ as

$$\text{Tr}(\sigma^2) = k_{(\mu;\nu)} k^{\mu;\nu} - \frac{1}{D-2} (k^\mu_{;\mu})^2, \quad (\text{A.6})$$

$$\theta = \frac{1}{D-2} k^\mu_{;\mu}, \quad (\text{A.7})$$

$$\text{Tr}(A^2) = -k_{[\mu;\nu]} k^{\mu;\nu}. \quad (\text{A.8})$$

Notice, in particular, that $\sigma_{ij} = 0 \Leftrightarrow \text{Tr}(\sigma^2) = 0$ and $A_{ij} = 0 \Leftrightarrow \text{Tr}(A^2) = 0$.

Using the decomposition (A.2) and eq. (A.5) it is also easy to see that

$$k_{[\mu;\nu]} k_{\rho]} = K_{i0} m_{[\mu}^{(i)} l_{\nu]} k_{\rho]} + A_{ij} m_{[\mu}^{(i)} m_{\nu]}^{(j)} k_{\rho]}, \quad (\text{A.9})$$

hence $k_{[\mu;\nu]} k_{\rho]} = 0$ if and only if k^μ is geodesic ($K_{i0} = 0$) and twistfree ($A_{ij} = 0$).

All the above is a natural extension of the geometric optics formalism developed and widely employed in $D = 4$ General Relativity [1–6, 8–10].

B. Rays of null Maxwell fields in higher dimensions

Given a “null” frame $(\mathbf{k}, \mathbf{l}, \mathbf{m}^{(i)})$ as in appendix A, the Maxwell 2-form $F_{\mu\nu}$ representing any electromagnetic field can be decomposed in terms of its frame components (of different boost weight) as $F_{\mu\nu} = 2F_{0i} l_{[\mu} m_{\nu]}^{(i)} + 2F_{01} l_{[\mu} k_{\nu]} + F_{ij} m_{[\mu}^{(i)} m_{\nu]}^{(j)} + 2F_{1i} k_{[\mu} m_{\nu]}^{(i)}$ [16, 24, 33]. We shall focus here on Maxwell fields that, in an appropriate frame, have only negative boost weight components, i.e. $F_{0i} = F_{01} = F_{ij} = 0$. Under this assumption the 2-form $F_{\mu\nu}$ takes the simple form

$$F_{\mu\nu} = 2F_{1i} k_{[\mu} m_{\nu]}^{(i)}, \quad (\text{B.1})$$

and (by analogy with the four-dimensional case [5, 6, 34–36]) it can then be referred to as a type N or *null* Maxwell field, k^μ being its multiple principal null direction. In fact, in any $D \geq 4$ dimension a Maxwell field has vanishing (zeroth-order) scalar invariants if and only if it is of type N [16].

Let us now consider consequences of the vacuum Maxwell equations $F^{\mu\nu}_{;\nu} = 0$ and $F_{[\mu\nu];\rho} = 0$ on optical properties of the aligned null direction k^μ . Using the decomposition (A.2), from the contractions $F^{\mu\nu}_{;\nu} k_\mu = 0$ and $F_{[\mu\nu];\rho} k^\mu m^{(i)\nu} m^{(j)\rho} = 0$ one finds, respectively,

$$F_{1i} K_{i0} = 0, \quad F_{1i} K_{j0} = F_{1j} K_{i0}. \quad (\text{B.2})$$

Contracting the second of eqs. (B.2) with K_{i0} and using the first of eqs. (B.2) it follows

$$K_{i0} = 0, \quad (\text{B.3})$$

i.e. \mathbf{k} is *geodesic*. Then, combining $F^{\mu\nu}_{;\nu} m_\mu^{(i)} = 0$ and $F_{[\mu\nu];\rho} l^\mu m^{(i)\nu} k^\rho = 0$ (and using $k^\mu_{;\mu} = K_{10} + (D-2)\theta$) one gets

$$2F_{1i} \sigma_{ij} - (D-4)\theta F_{1j} = 0. \quad (\text{B.4})$$

This equation has now somewhat different consequences in $D = 4$ and $D > 4$. For $D = 4$, one necessarily has $\sigma_{ij} = 0$ (this is most easily seen in a frame such that

$\sigma_{ij} = \text{diag}(\sigma_{22}, -\sigma_{22})$), i.e. the aligned null direction k^μ must be shearfree [35, 36] (irrespective of the value of θ). In contrast, for $D > 4$ if k^μ is expanding ($\theta \neq 0$) then it must be also shearing ($\sigma_{ij} \neq 0$).

It is worth remarking that aligned null directions of type N Weyl tensors display very similar properties [19].

References

- [1] R. Sachs, *Gravitational waves in general relativity. VI. The outgoing radiation condition*, *Proc. R. Soc. A* **264** (1961) 309–338.
- [2] J. Ehlers and W. Kundt, *Exact solutions of the gravitational field equations*, in *Gravitation: An Introduction to Current Research* (L. Witten, ed.), pp. 49–101. Wiley, New York, 1962.
- [3] E. Newman and R. Penrose, *An approach to gravitational radiation by a method of spin coefficients*, *J. Math. Phys.* **3** (1962) 566–578. See also E. Newman and R. Penrose (1963), Errata, *J. Math. Phys.* **4** (1963) 998.
- [4] F. A. E. Pirani, *Introduction to gravitational radiation theory*, in *Lectures on General Relativity* (S. Deser and K. F. Ford, eds.), vol. 1, pp. 249–373. Prentice-Hall, Englewood Cliffs, New Jersey, 1965.
- [5] R. Penrose and W. Rindler, *Spinors and Space-Time*, vol. 2. Cambridge University Press, Cambridge, 1986.
- [6] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein’s Field Equations*. Cambridge University Press, Cambridge, second ed., 2003.
- [7] I. Robinson and A. Trautman, *Spherical gravitational waves*, *Phys. Rev. Lett.* **4** (1960) 431–432.
- [8] I. Robinson and A. Trautman, *Some spherical gravitational waves in general relativity*, *Proc. R. Soc. A* **265** (1962) 463–473.
- [9] W. Kundt, *The plane-fronted gravitational waves*, *Z. Phys.* **163** (1961) 77–86.
- [10] W. Kundt, *Exact solutions of the field equations: twist-free pure radiation fields*, *Proc. R. Soc. A* **270** (1962) 328–332.
- [11] J. N. Goldberg and R. K. Sachs, *A theorem on Petrov types*, *Acta Phys. Polon. Suppl.* **22** (1962) 13–23.
- [12] A. Coley, R. Milson, N. Pelavas, V. Pravda, A. Pravdová, and R. Zalaletdinov, *Generalizations of pp-wave spacetimes in higher dimensions*, *Phys. Rev. D* **67** (2003) 104020.
- [13] Y. N. Obukhov, *Generalized plane-fronted gravitational waves in any dimension*, *Phys. Rev. D* **69** (2004) 024013.
- [14] A. Coley, S. Hervik, and N. Pelavas, *On spacetimes with constant scalar invariants*, *Class. Quantum Grav.* **23** (2006) 3053–3074.
- [15] A. Coley, A. Fuster, S. Hervik, and N. Pelavas, *Higher dimensional VSI spacetimes*, *Class. Quantum Grav.* **23** (2006) 7431–7444.
- [16] A. Coley, R. Milson, V. Pravda, and A. Pravdová, *Vanishing scalar invariant spacetimes in higher dimensions*, *Class. Quantum Grav.* **21** (2004) 5519–5542.

- [17] J. Podolský and M. Ortaggio, *Robinson-Trautman spacetimes in higher dimensions*, *Class. Quantum Grav.* **23** (2006) 5785–5797.
- [18] V. P. Frolov and D. Stojković, *Particle and light motion in a space-time of a five-dimensional rotating black hole*, *Phys. Rev. D* **68** (2003) 064011.
- [19] V. Pravda, A. Pravdová, A. Coley, and R. Milson, *Bianchi identities in higher dimensions*, *Class. Quantum Grav.* **21** (2004) 2873–2897.
- [20] J. Lewandowski and T. Pawłowski, *Quasi-local rotating black holes in higher dimension: geometry*, *Class. Quantum Grav.* **22** (2005) 1573–1598.
- [21] H. W. Brinkmann, *Einstein spaces which are mapped conformally on each other*, *Math. Ann.* **94** (1925) 119–145.
- [22] C. Böhm, *Inhomogeneous Einstein metrics on low-dimensional spheres and other low-dimensional spaces*, *Invent. Math.* **134** (1998) 145–176.
- [23] A. Coley, R. Milson, V. Pravda, and A. Pravdová, *Classification of the Weyl tensor in higher dimensions*, *Class. Quantum Grav.* **21** (2004) L35–L41.
- [24] R. Milson, A. Coley, V. Pravda, and A. Pravdová, *Alignment and algebraically special tensors in Lorentzian geometry*, *Int. J. Geom. Meth. Mod. Phys.* **2** (2005) 41–61.
- [25] F. R. Tangherlini, *Schwarzschild field in n dimensions and the dimensionality of space problem*, *Nuovo Cimento* **27** (1963) 636–651.
- [26] D. Birmingham, *Topological black holes in anti-de Sitter space*, *Class. Quantum Grav.* **16** (1999) 1197–1205.
- [27] G. W. Gibbons, D. Ida, and T. Shiromizu, *Uniqueness and nonuniqueness of static black holes in higher dimensions*, *Phys. Rev. Lett.* **89** (2002) 041101.
- [28] G. Gibbons and S. A. Hartnoll, *Gravitational instability in higher dimensions*, *Phys. Rev. D* **66** (2002) 064024.
- [29] H. W. Brinkmann, *Riemann spaces conformal to Einstein spaces*, *Math. Ann.* **91** (1924) 269–278.
- [30] L. P. Eisenhart, *Riemannian Geometry*. Princeton University Press, Princeton, second ed., 1949.
- [31] A. Z. Petrov, *Einstein Spaces*. Pergamon Press, Oxford, translation of the 1961 Russian edition, 1969.
- [32] V. Pravda, A. Pravdová, and M. Ortaggio, in preparation.
- [33] R. Milson, *Alignment and the classification of Lorentz-signature tensors*, [gr-qc/0411036](#).
- [34] J. L. Synge, *Relativity: the Special Theory*. North-Holland, Amsterdam, 1955.
- [35] L. Mariot, *Le champ électromagnétique singulier*, *C. R. Acad. Sci. Paris* **238** (1954) 2055–2056.
- [36] I. Robinson, *Null electromagnetic fields*, *J. Math. Phys.* **2** (1961) 290.